Interest rates, bond markets, derivatives

slides for the course *"Interest rate theory"*,
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part I

József Gáll
University of Debrecen

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Bond markets

In this course we shall discuss models of bond market over a finite time interval $[0, T^*]$, where $T^* \in \mathbb{R}$ is the time horizon, if not stated otherwise. (We note that most of the results can be generalised under an infinite time horizon.)

Bond contracts

- A bond is a financial asset, a debt security: the issuer of the bond is obliged to pay the holder of the bond the principal value (or face value) at maturity and possibly (depending on the contract) some interest (coupon payment) with a certain frequency (e.g. annually, semiannually).
- In a regular case, the coupon payment is given by either a fixed interest rate d, in which case the annual payment is dV, where V is the principal value. Or it can be a floating rate (e.g. LIBOR + x%).
- Zero coupon bond: a bond which has no coupon payment, hence the issuer is obliged to pay only the principal value at maturity.

Bond contracts (cont.)

- Given $T \in [0, T^*]$, a T-bond is defined to be a zero coupon bond with principal value of 1 unit of currency and with maturity date T.
- Notation: the value of the T-bond at time t is denoted by $P(t, T)$, and hence $P(T, T) = 1$, $0 \leq t \leq T \leq T^*$.
- Note that any fixed rate bond can be considered (i.e. equivalent) as a portfolio of zero coupon bonds. That is why we shall only use zero coupon bonds in the market models.
- Terminology. Non-defaultable bond: where the corresponding payments (coupon, principal) are payed with probability 1, i.e. there is no credit risk (default) assumed. Otherwise a bond is said to be defaultable.

Compounding conventions: the classical one

- Assume that we invest an amount X in an asset (e.g. bank account) such that it is compounded k times a year at rate r according to the "classical" compounding convention.
- In this case the value of our investment in a year time will be $X(1+\frac{r}{k})^k$, or in T years time $X(1+\frac{r}{k})^{Tk}$.
- Conversely, in order to realise Y at the end of year T we need to invest it's discounted value according to the classical compounding, i.e. an amount of $Y/(1+\frac{r}{k})^{Tk}$.

Compounding conventions: the continuous one

- Note that clearly $\lim_{k\to\infty} X(1+\frac{r}{k})^{Tk} = Xe^{rT}$. This gives the idea of continuous compounding.
- Assume that we invest an amount X in an asset (e.g. bank account) such that it is compounded at rate r according to the "continuous" compounding convention.
- In this case the value of our investment in a year time is defined to be Xe^{r}, or in T years time Xe^{rT}.
- Conversely, in order to realise Y at the end of year T we need to invest it's discounted value according to the continuous compounding, i.e. an amount of $Y/e^{rT} = Ye^{-rT}$.

Compounding conventions

It is obvious that given a rate r with k times a year classical compounding we can always find an equivalent rate $\tilde{r}$ according to the continuous compounding convention in the sense

$$\left(1 + \frac{r}{k}\right)^k = e^{\tilde{r}}.$$

Obviously $\tilde{r} = \ln\left((1 + \frac{r}{k})^k\right)$.

Rates of return

- Consider an asset S with values S_0 and S_T at time 0 and at time T (in T years time), respectively.
- Than, denoting the annual rate of return h of the asset according to the classical k times a year compounding, we have $S_T = S_0(1 + \frac{h}{k})^k$ and thus $h = ((S_T/S_0)^{1/(Tk)} - 1)k$ (for simplicity, assume that Tk is integer).
- On the other hand, for the annual rate of return $\tilde{h}$ of the asset according to the continuous compounding we have $S_T = S_0 e^{\tilde{h}T}$ and hence $\tilde{h} = \frac{1}{T}\ln(S_1/S_0)$, which is the so-called logreturn.

Day-count convention

- Given $0 \leq t \leq T \leq T^*$, the year fraction between dates t and T will be denoted by $\tau(t, T)$.
- A natural way of defining the value of it is $\tau(t, T) = T - t$, however this is not the only convention followed in the markets.
- If not stated otherwise, we shall assume that $\tau(t, T) = T - t$.

Further day-count conventions

- Actual/365: $\tau(t,T) = \frac{D_2 - D_1}{365}$, where $D_2 - D_1$ is the number of actual days between t and T (s.t. the day D_1 of t is included, but D_2 of T is excluded), hence, a year is assumed to have 360 (relevant) days.
- Actual/360: $\tau(t,T) = \frac{D_2 - D_1}{360}$, i.e. a year is assumed to have 360 (relevant) days.
- Denote the date of D_i by (d_i, m_i, y_i), $i = 1, 2$, in a date structure of $(\textit{day number}, \textit{month number}, \textit{year number})$. Then

$$\tau(t,T) =$$

$$\frac{360(y_2 - y_1) + 30(m_2 - m_1 - 1) + \max(30 - d_1, 0) + \min(d_2, 30)}{360}.$$

Spot interest rates I

Continuously-compounded spot interest rate at time t corresponding to maturity T:

$$R(t,T) := -\frac{\ln P(t,T)}{\tau(t,T)},$$

which is the (annual) rate according to continuous compounding embedded in the T-bond price, i.e.

$$P(t,T) = e^{-R(t,T)\tau(t,T)}.$$

Spot interest rates II

Simply-compounded spot interest rate at time t corresponding to maturity T:

$$L(t,T) := -\frac{1-P(t,T)}{\tau(t,T)P(t,T)},$$

which is the (annual) 'simple' rate embedded in the T-bond price, i.e.

$$P(t,T) = \frac{1}{1+L(t,T)\tau(t,T)}.$$

Spot interest rates III

Annually-compounded spot interest rate at time t corresponding to maturity T:

$$Y(t,T) := -\frac{1}{P(t,T)^{1/\tau(t,T)}} - 1,$$

which is the (annual) rate according to classical (ones-a-year) compounding embedded in the T-bond price, i.e.

$$P(t,T) = \frac{1}{[1+Y(t,T)]^{\tau(t,T)}}.$$

Spot interest rates IV

k-**times-a-year-compounded spot interest rate** at time t corresponding to maturity T:

$$Y_k(t,T) := -k\left(\frac{1}{P(t,T)^{1/[k\tau(t,T)]}} - 1\right) = -\frac{k}{P(t,T)^{1/[k\tau(t,T)]}} - k,$$

which is the (annual) rate according to classical (k-times-a-year) compounding embedded in the T-bond price, i.e.

$$P(t,T) = \frac{1}{\left[1 + \frac{Y_k(t,T)}{k}\right]^{k\tau(t,T)}}.$$

Interest rate curves

Yield curve or **zero-coupon curve** at time t is the function

$$T \mapsto \begin{cases} L(t,T) & \text{for } t < T \leq t+1, \\ Y(t,T) & \text{for } t+1 < T \leq T^*. \end{cases}$$

Yield curve or **zero-coupon curve** with continuous compounding at time t is the function

$$T \mapsto R(t,T), \quad T \in [t, T^*].$$

Zero bond curve at time t is the function

$$T \mapsto P(t,T), \quad T \in [t, T^*].$$

Forward interest rates

Simply-compounded forward interest rate at time t corresponding to the time interval T, S, where $0 \leq t \leq T \leq S \leq T^*$:

$$F(t,T,S) := \frac{1}{\tau(T,S)} \left(\frac{P(t,T)}{P(t,S)} - 1 \right),$$

which is the simple (annual) rate embedded in the T-bond price for the interval $[T, S]$, i.e.

$$P(t,S) = P(t,T) \frac{1}{1 + F(t,T,S)\tau(T,S)}.$$

Forward interest rates II

Continuously-compounded forward interest rate at time t corresponding to the time interval T, S, where $0 \leq t \leq T \leq S \leq T^*$:

$$\tilde{F}(t, T, S) := -\frac{1}{\tau(T, S)} \ln \frac{P(t, S)}{P(t, T)} = \frac{1}{\tau(T, S)} \ln \frac{P(t, T)}{P(t, S)},$$

which is the (annual) rate according to continuous compounding embedded in the T-bond price for the interval $[T, S]$, i.e.

$$P(t, S) = P(t, T) e^{-\tilde{F}(t, T, S)\tau(T, S)}.$$

Instantaneous forward interest rates

Assume that the zero bond curve is differentiable w.r.t. the maturity, and take day-count convention $\tau(t,T) = T - t$. Then

$$\lim_{S \to T+} F(t,T,S) = -\lim_{S \to T+} \frac{1}{P(t,S)} \frac{P(t,S) - P(t,T)}{S-T}$$

$$= -\frac{1}{P(t,T)} \frac{\partial P(t,T)}{\partial T} = -\frac{\partial \ln P(t,T)}{\partial T}.$$

The instantaneous forward interest rate at time t corresponding to maturity T is

$$f(t,T) := \lim_{S \to T+} F(t,T,S) = -\frac{\partial \ln P(t,T)}{\partial T},$$

and hence we have

$$P(t,T) = \mathrm{e}^{-\int_t^T f(t,u)\, du}.$$

Instantaneous forward interest rates II

Similarly, one can obtain the same notion from the continuously-compounded forward rate:

$$\lim_{S\to T+} \tilde{F}(t, T, S) := \lim_{S\to T+} -\frac{\ln P(t,S) - \ln P(t,T)}{S - T}$$

$$= -\frac{\partial \ln P(t,T)}{\partial T} = -\frac{1}{P(t,T}\frac{\partial P(t,T)}{\partial T}$$

$$= f(t,T).$$

FRA

Let $K(>0)$ be a fixed interest rate, $0 \leq t < T < S \leq T^*$.
A **Forward rate agreement (FRA)** provides it's owner an exchange of a fixed interest payment K against a floating interest payment based on the spot rate $L(T,S)$ over the time interval $[T,S]$. At t the agreement is set up, with expiry date T with a nominal value N (i.e. the number of units of currency over which the interest are payed). At maturity S the payoff received by the holder of the contract is

$$N\tau(T,S)(K - L(T,S)),$$

provided that the same day-count convention is applied to both interest rates.

FRA vs forward rate

Consider the FRA of the previous slide. The payoff at time S is

$$N\tau(T,S)(K-L(T,S)) = N\tau(T,S)\left(K - \frac{1}{P(T,S)} + 1\right).$$

Investing $P(t,S)N\tau(T,S)(K+1)$ at time t in S-bond gives $N\tau(T,S)(K+1)$ at time S, similarly, investing $N\tau(T,S)P(t,T)$ in T-bond at time t gives 1 unit of currency at T, which can be reinvested in S-bond, by buying $N\tau(T,S)/P(T,S)$ number of S-bond, which provides value $N\tau(T,S)/P(T,S)$ at time S. Hence the value of the FRA at time t must be

$$P(t,S)N\tau(T,S)(K+1) - P(t,T) = P(t,S)N(K - F(t,T,S)).$$

Thus K should be chosen to be equal to $F(t,T,S)$ to set the value of the FRA zero at time t.

Notations

From now on we shall use the following notations for the definitions of further interest derivatives.
Let

- t be the current time and
- $[T_\alpha, T_\beta]$ a future time interval divided into sub-intervals: $0 \leq t < T_\alpha < T_{\alpha+1} < \ldots < T_{\alpha+n} = T_\beta \leq T^*$,
- N is the nominal value of the contracts,
- K is a fixed interest rate.

Interest rate swap (IRS), RFS

Receiver IRS or **Receiver (Forward-start) interest rate swap (RFS)**: it is a contract which provides it's holder a sequence of exchanges of fixed interest payments K against the floating interest payments based on the spot rate $L(T_{\alpha+i}, T_{\alpha+i+1})$ over the time interval $[T_{\alpha+i}, T_{\alpha+i+1}]$, based on the nominal value N, for all $i = 0, 1, \ldots, n-1$. At t the agreement is set up, at $T_{\alpha+i+1}$ the payoff

$$N\tau(T_{\alpha+i}, T_{\alpha+i+1})(K - L(T_{\alpha+i}, T_{\alpha+i+1}))$$

is received by the holder of the position, for $i = 0, 1, \ldots, n-1$. Hence, the RFS is a sequence of FRA agreements.

Interest rate swap (IRS), RFS

Payer IRS or **Payer (Forward-start) interest rate swap (PFS)**: the position of the other party of the RFS, i.e. it is a contract which provides the holder of the position a sequence of floating interest payments based on the spot rate $L(T_{\alpha+i}, T_{\alpha+i+1})$ against the fixed interest payments K over the time interval $[T_{\alpha+i}, T_{\alpha+i+1}]$, based on the nominal value N, for all $i = 0, 1, \dots, n-1$. At t the agreement is set up, at $T_{\alpha+i+1}$ the payoff

$$N\tau(T_{\alpha+i}, T_{\alpha+i+1})(L(T_{\alpha+i}, T_{\alpha+i+1}) - K)$$

is received by the holder, for $i = 0, 1, \dots, n-1$.

Caplet

Let i be fixed.
A **caplet** with expiry date $T_{\alpha+i}$ and maturity $T_{\alpha+i+1}$ gives it's owner a payoff

$$N\tau(T_{\alpha+i}, T_{\alpha+i+1})(L(T_{\alpha+i}, T_{\alpha+i+1}) - K)^+$$

at time $T_{\alpha+i+1}$.
Suppose that we are supposed to pay an interest payment at the floating rate $L(T_{\alpha+i}, T_{\alpha+i+1})$ over the time interval $[T_{\alpha+i}, T_{\alpha+i+1}]$. If we hold a caplet above then it gives us the right to pay the interest at most at interest rate K.

Floorlet

Let i be fixed.
A **floorlet** with expiry date $T_{\alpha+i}$ and maturity $T_{\alpha+i+1}$ gives it's owner a payoff

$$N\tau(T_{\alpha+i}, T_{\alpha+i+1})(K - L(T_{\alpha+i}, T_{\alpha+i+1}))^{+}$$

at time $T_{\alpha+i+1}$.
Suppose that we are to receive an interest payment at the floating rate $L(T_{\alpha+i}, T_{\alpha+i+1})$ over the time interval $[T_{\alpha+i}, T_{\alpha+i+1}]$. If we hold a floorlet above then it gives us the right to receive an interest payment at least at interest rate K.

Cap and floor

A cap is a sequence of caplets.
A **cap** gives it's owner the sequence of payoffs

$$N\tau(T_{\alpha+i}, T_{\alpha+i+1})(L(T_{\alpha+i}, T_{\alpha+i+1}) - K)^+$$

at time points $T_{\alpha+i+1}$, where $i = 0, 1, \ldots, n-1$.
A **floor** gives it's owner the sequence of payoffs

$$N\tau(T_{\alpha+i}, T_{\alpha+i+1})(K - L(T_{\alpha+i}, T_{\alpha+i+1}))^+$$

at time points $T_{\alpha+i+1}$, where $i = 0, 1, \ldots, n-1$.

Swaption (Swap option)

- Let $T \in [t, T_\alpha]$ be the maturity of the swpation.
- A European **payer swaption** or **payer swap option** gives the right of the holder to enter a payer IRS (PFS) at time T.
- A European **receiver swaption** or **receiver swap option** gives the right of the holder to enter a receiver IRS (RFS) at time T.
- Note that the maturity of such contracts is often the first reset date of the IRS, i.e. T_α.
- The swaption cannot be decomposed in more elementary contracts as it was the case with caps and floors (decomposed in caplets and floorlets).

Bibliographic notes

For the basic notion discussed in this part of the course we mainly used Chapters 1 of Brigo & Mercurio (2006), and Chapter 1 of Cairns (2004), however we note that one can find several books where the main notions and assets of interest rate and bond markets are discussed. We also refer to Chapter 9 of Musiela & Rutkowski (2005), furthermore, for some fundamental financial background we refer to the excellent monograph of Hull (2012), in particular see Chapters 4,6,7.

References

Brigo, D. and Mercurio, F. (2006), *Interest Rate Models - Theory and Practice: With Smile, Inflation and Credit*, Springer, Berlin Heidelberg New York.

Cairns, A. J. G. (2004), *Interest Rate Models - An introduction*, Princeton University Press, Princeton and Oxford.

Hull, J. C (2012), *Options, Futures, and Other Derivatives*, Eighth Edition, Pearson Education Limited (Global Edition.).

Musiela, M. and Rutkowski, M. (2005), *Martingale Methods in Financial Modeling*, second edition, Springer-Verlag, Berlin, Heidelberg.