Interest rate models in continuous time

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General assumptions, notations

- We assume that we have a prob. space $(\Omega, \mathcal{F}, \mathbb{P})$ in the background with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T^*]}$, where $\mathbb{P}$ is the market measure.
- If not stated otherwise, we shall assume that W is a d-dimensional std. Brownian motion on the space such that the filtration is the augmented version of the filtration generated by W.

Risk-free asset

- The short rate process r is supposed to be an adapted process with almost all sample paths integrable over $[0, T^*]$ a.s.
- The bank account (or the risk free asset) is defined by

$$B_t = e^{\int_0^t r_u \, du}, \quad t \in [0, T^*],$$

and hence we have $dB_t = r_t B_t dt$ with initial value $B_0 = 1$ for the function $t \mapsto B_t(\omega)$ for almost every $\omega \in \Omega$.

Bond prices

The bond price processes $P(\cdot, T)$ for all maturity $T \in [0, T^*]$ are assumed to be strictly positive adapted processes.

Definition. We call the set of bond prices $P(t, T)$, $0 \leq t \leq T \leq T^*$ an arbitrage-free family of bond prices relative to r if

- $P(T, T) = 1$ for all $T \in [0, T^*]$, and
- there exists a prob. measure $\mathbb{P}^*$ such that it is equivalent to $\mathbb{P}$ and the discounted bond price processes

$$Z(t, T) = \frac{P(t, T)}{B_t}, \quad t \in [0, T],$$

form $\mathbb{P}^*$-martingales for all maturity $T \in [0, T^*]$. (Hence $\mathbb{P}^*$ is an EMM.)

Bond prices under $\mathbb{P}^*$

Since $Z(t, T) = \mathbb{E}_{\mathbb{P}^*}(Z(T, T) \mid \mathcal{F}_t)$, $t \in [0, T]$, we have for the bond prices

$$P(t, T) = B_t \mathbb{E}_{\mathbb{P}^*}(B_T^{-1} \mid \mathcal{F}_t), \quad t \in [0, T],$$

or to put it another way

$$P(t, T) = \mathbb{E}_{\mathbb{P}^*}\left(\mathrm{e}^{-\int_t^T r_u \, du} \mid \mathcal{F}_t\right), \quad t \in [0, T]. \qquad (1)$$

Remark. One can define a bond market with an arbitrage-free family of bond prices such that one defines a spot rate process r and an equivalent prob. measure $\mathbb{P}^*$, and finally defines the bond prices by (1).

Derivative prices, summary of facts

Given a bond market model with arbitrage-free family of bond prices, assume that we have a contingent claim, say, H, with maturity $T \in [0, T^*]$, which is an r.v. measurable w.r.t. $\mathcal{F}_T$, such that it is attainable in the sense that there is a self-financing trading strategy constructed using a finite number of assets (bonds, bank account) of the market, which strategy has a value H at T a.s. In this case the unique arbitrage-free price of the claim shall be

$$\pi_t = \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_t^T r_u\, du} H \mid \mathcal{F}_t\right), \quad t \in [0, T]. \tag{2}$$

Derivative prices, summary of facts (cont.)

- We do not discuss the general theory of arbitrage and market completeness (e.g. characterisation in case of infinitely many assets).
- In what follows we shall only consider attainable claims, where the replicating strategy can be constructed using finite number of assets.

Assumptions

In this section we assume that we have a bond market with an arbitrage-free family of bond prices and the short rate process r is given by an Itô process, i.e.

$$dr_t = \mu_t \, dt + \sigma_t \, dW_t,$$

with initial value $r_0 > 0$, where μ and σ are adapted $\mathbb{R}$ ans $\mathbb{R}^d$ valued adapted processes s.t. r is well defined, furthermore W is a d-dimensional std. Brownian motion, and the filtration is the augmented version of the one generated by W.

Notes on $\mathbb{P}^*$.

We know in this case that for an EMM

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \varepsilon_{T^*}\left(\int_0^{\cdot} \lambda_u \, dW_u\right)$$

with an appropriate adapted process λ, furthermore, writing $\eta_{T^*} := \frac{d\mathbb{P}^*}{d\mathbb{P}}$ and the density process η is given by $\eta_t = \mathbb{E}_{\mathbb{P}}(\eta_{T^*} \mid \mathcal{F}_t)$ a.s., and we have

$$\eta_t = e^{\int_0^t \lambda_u \, dW_u - \frac{1}{2}\int_0^t |\lambda_u|^2 \, du},$$

by the Girsanov theorem we know that

$$W_t^* := W_t - \int_0^t \lambda_u \, du, \quad t \in [0, T^*],$$

is a std. Brownian motion w.r.t. $\mathbb{P}^*$.

Bond price under an EMM $\mathbb{P}^*$.

Theorem. Under the above assumptions

- The short rate process under an EMM $\mathbb{P}^*$ is of the form

$$dr_t = \mu_t + \sigma_t \lambda_t \, dt + \sigma_t \, dW_t^*.$$

- for the bond price with maturity $T \in [0, T^*]$ there is a $\mathbb{R}^d$ valued process $b^\lambda(\cdot, T)$ such that under $\mathbb{P}^*$

$$dP(t, T) = P(t, T) r_t \, dt + P(t, T) b^\lambda(t, T) \, dW_t^*, \qquad (3)$$

and

$$P(t, T) = P(0, T) B_t \varepsilon_t \left(\int_0^{\cdot} b^\lambda(u, T) \, dW_u^* \right).$$

Volatility of bonds

Remark. Due to the Girsanov theorem the process $b^\lambda(\cdot, T)$ does not depend on the choice of the equivalent (martingale) measure, which leads to the next notion.

Definition. The process $b^\lambda(\cdot, T)$ is called the volatility of the T-bond (i.e. of the zero coupon bond with maturity T), $T \in [0, T^*]$.

Bond price under an equivalent measure

Theorem. Under the above assumptions assume that $\bar{\mathbb{P}}$ is another prob. measure such that it is equivalent to an EMM $\mathbb{P}^*$ (and hence to $\mathbb{P}$). Then for the bond price under $\bar{\mathbb{P}}$ we have

$$P(t,T) = \mathbb{E}_{\bar{\mathbb{P}}}\left(\mathrm{e}^{-\int_t^T r_u \, du} \mathrm{e}^{\int_t^{T^*} (\lambda_u - \bar{\lambda}_u) \, d\bar{W}_u - \frac{1}{2} \int_t^{T^*} |\lambda_u - \bar{\lambda}_u|^2 \, du} \mid \mathcal{F}_t \right)$$

a.s. for $t \in [0, T]$, where $\bar{\lambda}$ is the adapted process corresponding to $\bar{\mathbb{P}}$ according to

$$\frac{d\bar{\mathbb{P}}}{d\mathbb{P}} = \varepsilon_{T^*}\left(\int_0^{\cdot} \bar{\lambda}_u \, dW_u \right).$$

Remark. $\bar{\mathbb{P}}$ can be for instance a forward measure (see next section).

Market price of risk

Due to the above theorems, especially, (3), under the market measure $\mathbb{P}$ we have

$$dP(t,T) = P(t,T)r_t - b^\lambda(t,T)\lambda_t\ dt + P(t,T)b^\lambda(t,T)\ dW_t.$$

The extra term appearing in the drift, $-b^\lambda(t,T)\lambda_t$, is called the market price of risk or risk premium.

Matching the initial yield curve

In arbitrage free family of bond prices discussed in this section we have for the initial bond values

$$P(0,T) = \mathbb{E}_{\mathbb{P}^*}\left(e^{-\int_0^T r_u\, du}\right), \quad T \in [0, T^*],$$

under an EMM $\mathbb{P}^*$. This gives a special requirement when fitting the models to real data. In many models, especially in simple short rate models (see next section) matching the initial bond prices is not satisfied. This is an important motivation to introduce the forward interest rate models, e.g. the Heath-Jarrow-Morton type models.

A general idea of change of numeraire

First we study the general idea of change of numeraire. Let us consider a continuous time market of finitely many assets. Let $\mathbb{P}$ be the market measure. Assume that there is an asset (without dividend) with positive price process N, and $\mathbb{P}_N$ is a prob. measure, equivalent to $\mathbb{P}$ such that the price process X of any asset discounted by the numeraire forms a $\mathbb{P}_N$ martingale, that is

$$\frac{X_t}{N_t} = \mathbb{E}_{\mathbb{P}_N}\left(\frac{X_T}{N_T} \mid \mathcal{F}_t\right) \quad \text{a.s. for } 0 \leq t \leq T \leq T^*.$$

A general idea of change of numeraire (cont.)

Assume now that U is the price process of another asset, with positive values a.s. Then it is easy to show that there exists a prob. measure $\mathbb{P}_U$, which is equivalent to $\mathbb{P}$ (and hence to $\mathbb{P}_N$) such that the price process X of any asset discounted by the numeraire forms a $\mathbb{P}_N$ martingale, that is

$$\frac{X_t}{U_t} = \mathbb{E}_{\mathbb{P}_U}\left(\frac{X_T}{U_T} \mid \mathcal{F}_t\right) \quad \text{a.s. for } 0 \leq t \leq T \leq T^*.$$

Indeed, observe that choosing

$$\frac{d\mathbb{P}_U}{d\mathbb{P}_N} = \frac{U_T^* N_0}{U_0 N_T^*}$$

to be the Radon-Nikodym derivative of the new measure will lead to the desired statement by applying the abstract Bayes formula.

Forward measures

Definition. Let $T \in [0, T^*]$ be a fixed maturity and consider the T-bond price process $P(\cdot, T)$. Take the restricted measurable space $(\Omega, \mathcal{F}_T)$, and define the prob. measure $\mathbb{P}_T$ equivalent to $\mathbb{P}^*$ (and hence to $\mathbb{P}$) by

$$\frac{d\mathbb{P}_T}{d\mathbb{P}^*} := \frac{P(T,T)B_0}{P(0,T)B_T} = \frac{1}{P(0,T)B_T} = \frac{B_T^{-1}}{\mathbb{E}_{\mathbb{P}^*} B_T^{-1}}, \quad a.s.$$

Then $\mathbb{P}_T$ is called the T-forward measure, or the forward (martingale) measure for the settlement date T.

Forward price of a claim using $\mathbb{P}^*$

Theorem. Assume that X is an attainable claim corresponding to settlement date T (thus X is $\mathcal{F}_T$-measurable). Let F_t denote the forward price of X at time t, $t \in [0, T]$. Then we have

$$F_t = \frac{\mathbb{E}_{\mathbb{P}^*}(XB_T^{-1} \mid \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}^*}(B_T^{-1} \mid \mathcal{F}_t)} = \frac{\pi_t}{P(t, T)},$$

where π_t is the arbitrage-free price of claim X at time t.
Remark. If someone writes a forward contract at t on X with maturity T (of the forward contract) then the forward price F_t will be the delivery price of this contract.

Forward price of a claim using $\mathbb{P}_T$

Theorem. Under the assumption of the previous theorem the forward price F_t of an attainable claim X with settlement date T is

$$F_t = \mathbb{E}_{\mathbb{P}_T}(X \mid \mathcal{F}_t), \quad t \in [0, T],$$

where π_t is the arbitrage-free price of claim X at time t.

Forward price of a claim using $\mathbb{P}_T$

Theorem. Assume that X is an attainable claim corresponding to settlement date U, and let π_t denote the arbitrage-free price of it at time t. Then

- using $\mathbb{P}_T$, where $T = U$, we have

$$\pi_t = P(t,T)\,\mathbb{E}_{\mathbb{P}_T}(X \mid \mathcal{F}_t), \quad t \in [0,T],$$

- using $\mathbb{P}_T$, where $U \le T \le T^*$, we have

$$\pi_t = P(t,T)\,\mathbb{E}_{\mathbb{P}_T}(XP^{-1}(U,T) \mid \mathcal{F}_t), \quad t \in [0,U],$$

- using $\mathbb{P}_T$, where $0 \le T \le U$, we have

$$\pi_t = P(t,T)\,\mathbb{E}_{\mathbb{P}_T}(XP(T,U) \mid \mathcal{F}_t), \quad t \in [0,T].$$

Simply-compounded forward rate process

Theorem. The simply-compounded forward rate $F(\cdot, S, T)$, corresponding to the time interval $[S, T]$ $(0 \le S \le T \le T^*)$ is a martingale under the T-forward measure $\mathbb{P}_T$, that is

$$F(u, S, T) = \mathbb{E}_{\mathbb{P}_T}(F(t, S, T) \mid \mathcal{F}_u), \quad \text{a.s. } 0 \le u \le t \le S,$$

in particular

$$F(u, S, T) = \mathbb{E}_{\mathbb{P}_T}(F(S, S, T) \mid \mathcal{F}_u) = \mathbb{E}_{\mathbb{P}_T}(L(S, T) \mid \mathcal{F}_u) \quad \text{a.s.},$$

$0 \le u \le S$, where $L(S, T)$ is the simply-compounded spot rate corresponding to maturity T.

On an expectation hypothesis

Theorem. The instantaneous forward rate $f(t, T)$ is the expectation of the spot rate r_T under the T-forward measure, more precisely

$$f(t, T) = \mathbb{E}_{\mathbb{P}_T}(r_T \mid \mathcal{F}_t), \quad 0 \leq t \leq T \leq T^*.$$

Basic assumptions

- Like in the previous section, we assume that we have a prob. space $(\Omega, \mathcal{F}, \mathbb{P})$ in the background with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T^*]}$, where $\mathbb{P}$ is the market measure.
- We assume, that there is an EMM $\mathbb{P}^*$, and the model is given directly under $\mathbb{P}^*$. For this, W^* shall be a std. Brownian motion under $\mathbb{P}^*$, if not stated otherwise, a one dimensional one.
- In this section we assume that the short rate dynamics is given by a diffusion process.
- Terminology: based on the "number of sources of uncertainty" we have one-factor (single-factor) , and multi-factor models.

Merton's model

Assume that the short rate process r is given by

$$r_t = r_0 + at + \sigma W_t^*, \quad t \in [0, T^*],$$

where a, σ, r_0 are positive constants.
Theorem. Under no arbitrage we have for the zero coupon bond prices

$$P(t, T) = e^{-r_t(T-t) + -\frac{1}{2}a(T-t)^2 + \frac{1}{6}\sigma^2(T-t)^3}, \quad 0 \leq t \leq T \leq T^*,$$

furthermore, under $\mathbb{P}^*$

$$dP(t, T) = P(t, T) r_t dt - P(t, T)\sigma(T - t) dW_t^*.$$

Vasicek's model

In this model we assume that the short rate process r is given by

$$dr_t = (a - br_t)dt + \sigma dW_t^*, \tag{4}$$

where a, b, and σ are positive constants.

Theorem. The short rate process takes the form

$$r_t = r_s e^{-b(t-s)} + \frac{a}{b}\left(1 - e^{-b(t-s)}\right) + \sigma \int_s^t e^{-b(t-s)} dW^*,$$

(which is the unique solution of (4)) , hence the conditional distribution of r_t w.r.t. $\mathcal{F}_s$ is normal (Gaussian) with conditional expectation

$$\mathbb{E}_{\mathbb{P}^*}(r_t|\mathcal{F}_t) = r_s e^{-b(t-s)} + \frac{a}{b}\left(1 - e^{-b(t-s)}\right)$$

Vasicek's model (cont.)

and conditional variance

$$\mathrm{Var}_{\mathbb{P}^*}(r_t|\mathcal{F}_s) = \frac{\sigma^2}{2b}\left(1 - \mathrm{e}^{-2b(t-s)}\right).$$

Remark. Clearly

$$\lim_{t\to\infty} \mathbb{E}_{\mathbb{P}^*}(r_t|\mathcal{F}_s) = \frac{a}{b} \quad \text{and} \quad \lim_{t\to\infty} \mathrm{Var}_{\mathbb{P}^*}(r_t|\mathcal{F}_s) = \frac{\sigma^2}{2b}.$$

Bond prices in the Vasicek's model

Theorem. Under no arbitrage we have for the zero coupon bond prices

$$P(t, T) = e^{m(t,T) - n(t,T) r_t},$$

where

$$n(t, T) = \frac{\sigma^2}{2}\left(1 - e^{-b(t-s)}\right),$$

$$m(t, T) = \frac{\sigma^2}{2} \int_t^T n^2(u, T)\, du - a \int_t^T n(u, T)\, du, \quad 0 \le t \le T \le T^*.$$

furthermore, the volatility of $P(t, T)$ is $b(t, T) = -\sigma n(t, T)$ and thus under $\mathbb{P}^*$

$$dP(t, T) = P(t, T) r_t dt - P(t, T) \sigma n(T - t) dW_t^*.$$

Remarks.

- For derivative pricing it may be sufficient to introduce the model and to work only under $\mathbb{P}^*$, hence this way the risk premium does not appear in the formulas.
- For other problems, however, one should also use the market measure (e.g. risk measures calculations, or in case of historical type of estimations instead of calibration methods).
- Vasicek's model has a mean reverting property (see the drift of r).

Further short rate models

- Dothan's model: $dr_t = \sigma r_t dW_t^*$, where σ is a positive constant. By the Itô formula one can show that $r_t = r_0 e^{\sigma W^* - \frac{1}{2}\sigma^2 t}$, for $t \in [0, T^*]$, and hence the short rate has lognormal distribution under the EMM.
- Cox-Ingersoll-Ross model: $dr_t = (a - br_t)dt + \sigma\sqrt{r_t}dW_t^*$, where a, b and σ are positive constants. Also known as the "square-root process".
- Hull-White model, the general time-inhomogeneous version: $dr_t = (a_t - b_t r_t)dt + \sigma_t r_t^\beta dW_t^*$, where β is a non-negative constant, and the functions a, b, $\sigma : \mathbb{R}_+ \to \mathbb{R}$ are locally bounded. Note: much easier to fit to the initial term structure due to inhomogeneity.

The Heath-Jarrow-Morton (HJM) model, basic assumption

- Assume that we have a prob. space $(\Omega, \mathcal{F}, \mathbb{P})$ in the background with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T^*]}$, where $\mathbb{P}$ is the market measure.
- Let W be a d-dimensional std. Brownian motion. Suppose that the filtration is the augmented version of the filtration generated by W.
- We assume that the dynamics of the instantaneous forward rates $f(t,T)$ are given by

$$f(t,T) = f(0,T) + \int_0^T \alpha(u,T)du + \int_0^T \sigma(u,T)dW_u \quad t \in [0,T^*],$$

or

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t$$

for every fixed $T \in [0,T^*]$, where

HJM assumptions (cont.)

- the function $f(0,\cdot) : [0, T^*] \to \mathbb{R}$, which is assumed to be Borel-measurable, gives the initial forward rate structure,
- $\alpha : C \times \Omega \to \mathbb{R}$, $\sigma : C \times \Omega \to \mathbb{R}^d$, with $C = \{(u,t) \mid 0 \le u \le t \le T^*\}$, such that
- for every fixed $T \in [0, T^*]$ the processes $\alpha(\cdot, T)$, $\sigma(\cdot, T)$ are adapted with

$$\int_0^T |\alpha(u, T)| du < \infty, \quad \int_0^T |\sigma(u, T)|^2 du < \infty$$

$\mathbb{P}$-a.s.

HJM assumptions (cont.)

Recall that

- $r_t = f(t, t),\ t \in [0, T^*]$,
- the bank account (or risk free asset) is given by $B_t = \exp(\int_0^t f(u, u)du),\ t \in [0, T^*]$, and
- the bond price is defined as $P(t, T) = e^{-\int_t^T f(t,u)du}, \quad t \in [0, T]$.

Bond price under $\mathbb{P}$ in HJM models

Theorem. The bond price dynamics in the HJM model is given by

$$dP(t,T) = P(t,T)a(t,T)dt + P(t,T)b(t,T)dW_t,$$

where

$$a(t,T) = f(t,t) - \alpha^*(t,T) + \frac{1}{2}|\sigma^*(t,T)|^2 \quad \text{and} \quad b(t,T) = -\sigma^*(t,T)$$

for $t \in [0,T]$ with

$$\alpha^*(t,T) = \int_t^T \alpha(t,u)du \quad \text{and} \quad \sigma^*(t,T) = \int_t^T \sigma(t,u)du.$$

Change of measure in HJM models

No-arbitrage conditions I. We assume that there exists an adapted process λ with values in $\mathbb{R}^d$ such that

$$\mathbb{E}_{\mathbb{P}}\left\{\varepsilon_{T^*}\left(\int_0^{\cdot}\lambda_u\,dW_u\right)\right\}=1$$

and for $0\le T\le T^*$ we have

$$\alpha^*(t,T)=\frac{1}{2}|\sigma^*(t,T)|^2-\sigma^*(t,T)\lambda_t,$$

which by differentiation gives

$$\alpha(t,T)=\sigma(t,T)(\sigma^*(t,T)-\lambda_t),\quad t\in[0,T].$$

Drift condition

Theorem. Under the above *No-arbitrage condition I* let

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \left\{ \varepsilon_{T^*} \left(\int_0^{\cdot} \lambda_u \, dW_u \right) \right\} \quad \text{a.s.}$$

Then $\mathbb{P}^*$ is an EMM and the bond prices satisfy

$$dP(t,T) = P(t,T) r_t dt - P(t,T)\sigma^*(t,T) dW_t^*,$$

where $W_t^* := W_t - \int_0^t \lambda_u \, du$, $t \in [0,T]$, is a Brownian motion w.r.t. $\mathbb{P}^*$ (given by the Girsanov theorem). Furthermore, for the forward rate we have

$$df(t,T) = \sigma(t,T)\sigma^*(t,T)dt + \sigma(t,T)dW_t^*,$$

and for the short rate

$$r_t = f(0,t) + \int_0^t \sigma(u,t) \, du + \int_0^t \sigma(u,t) \, dW_u^*.$$

Forward measure in HJM models

Condition II..
Assume that there exists an adapted process h with values in $\mathbb{R}^d$ such that

$$\mathbb{E}_{\mathbb{P}}\left\{\varepsilon_{T^*}\left(\int_0^{\cdot} h_u\, dW_u\right)\right\} = 1$$

and for $0 \leq T \leq T^*$ we have

$$\int_T^{T^*} \alpha(t,u)\, du = -\frac{1}{2}\left|\int_T^{T^*} \sigma(t,u)\, du\right|^2 - h_t\left|\int_T^{T^*} \sigma(t,u)\, du\right|,$$

which by differentiation (w.r.t. T) gives

$$\alpha(t,T) = -\sigma(t,T)\left(h_t + \int_T^{T^*} \sigma(t,u)\, du\right), \quad t \in [0,T].$$

Forward measure in HJM models (cont.)

Theorem. Under the above *Condition II* let

$$\frac{d\mathbb{P}^{T^*}}{d\mathbb{P}} = \left\{ \varepsilon_{T^*}\left(\int_0^{\cdot} h_u \, dW_u \right) \right\} \quad \text{a.s.}$$

Then $\mathbb{P}^{T^*}$ is the T^*-forward measure in the market, i.e. the processes $P(t,T)/P(t,T^*)$, $t \in [0,T]$, are martingale under $\mathbb{P}^{T^*}$ for every maturity $T \in [0,T^*]$.

Remark. It can also be shown that *No-arbitrage condition I* and *Condition II* are equivalent, which is based on the relationship

$$\lambda_t = h_t + \int_t^{T^*} \sigma(t,u) \, du, \quad t \in [0,T^*].$$

Bibliographic notes

In the discussion of continuous time models in the course we used Chapters 9-11 in Musiela & Rutkowski (2005) and Chapter 2 in Brigo & Mercurio (2006), although we did not cover the whole chapters. Note that the latter monograph contains a very detailed discussion of short and forward rate models (Part II). For more on arbitrage theory and on market price of risk we refer to Björk (1998). We also mention the work of Pelsser (2000) for a general discussion of interest rate derivative pricing, and Hull (2012), which gives an introduction to interest rate models with the discussion of related financial issues, in particular in Chapters 28-31.

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References II

Finally we give a list of some classical papers of the literature (suggested reading), mostly on forward rate models.

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